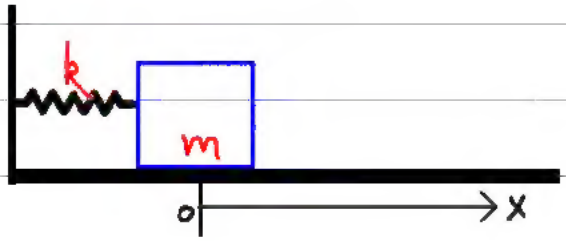
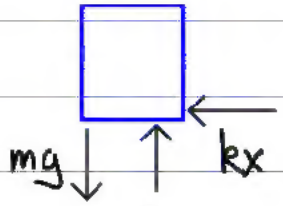
Meriam Chapter 8/2: Vibrations

Free vibrations:



Consider a mass m on a frictionless surface.
Spring is relaxed when m is at $x = 0$

→ $\Sigma \vec{F} = 0$ at $x = 0$



If mass is displaced a small +ve distance x, & then released

→ $\Sigma F_x = -kx$

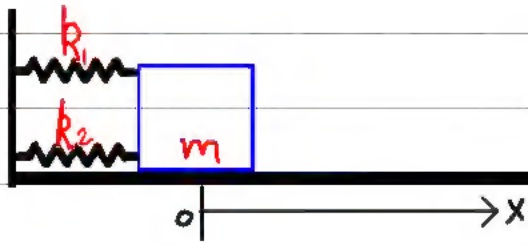
Note:

1/ F_x is linearly dependent on x → linear equation to describe vibration and details of motion.

2/ F_x is in opposite direction to displacement
→ F_x is a restoring force → stability

3/ If there is more than one spring, you can combine them into a single equivalent spring K. Place origin at the overall neutral position of the system

→ $F_x = -Kx$



governing equation (in one dimension):

$$\Sigma F_x = -kx = m\ddot{x}$$

or $$\boxed{m\ddot{x} + kx = 0}$$

Second order, linear, homogeneous, ordinary differential equation

Solution:

First put the equation into standard form:

$$\boxed{\ddot{x} + \omega_n^2 x = 0}$$

, $\omega_n = \sqrt{k/m}$

= natural frequency (s^{-1})

Solution has 2 undetermined constants:

$$\boxed{x(t) = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)}$$

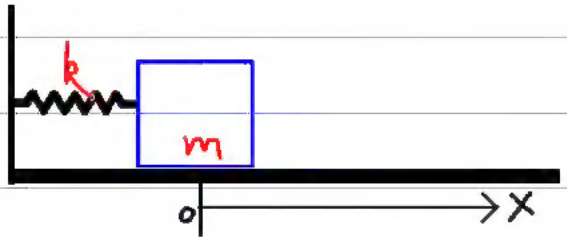
or $$x(t) = \boxed{C_3 \sin(\omega_n t + \phi)}$$

where $C_3 = \sqrt{C_1^2 + C_2^2}$ & $\phi = \tan(C_2/C_1)$

C_1 & C_2 (or C_3 & ϕ) depend on initial conditions.

$\omega_n = \sqrt{k/m}$ depends on system parameters k, m.

e.g.



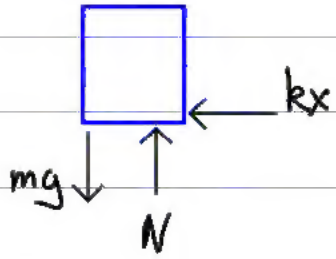
Given : $k = 10\ N/m$

$m = 5\ kg$

$x(t=0) = 3\ m$

$\dot{x}(t=0) = 0$

Find x(t) & describe motion

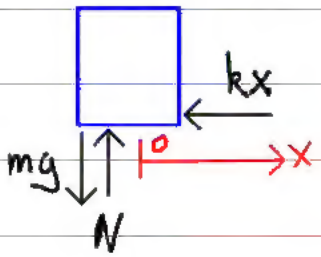


Solution :

1/ Define origin to be the location where all forces add to zero. This will make the arithmetic much easier.

2/ Choose a direction for the positive axis (if it was not already specified in the question

3/ Draw the FBD



4/ Displace the mass by a small +ve distance x from the origin, and then release it

5/ Sum the forces along the axis of displacement = $m\ddot{x}$

→ $$\Sigma F_x = -kx = m\ddot{x}$$

6/ Put the equation in standard form:

$\ddot{x} + \omega_n^2 x = 0$, where $\omega_n = \sqrt{k/m} = \sqrt{10/5} = \sqrt{2}\ s^{-1}$

7/ solution: $x(t) = C_3 \sin(\omega_n t + \phi)$ (ω_n in $\sqrt{2}\ s^{-1}$)

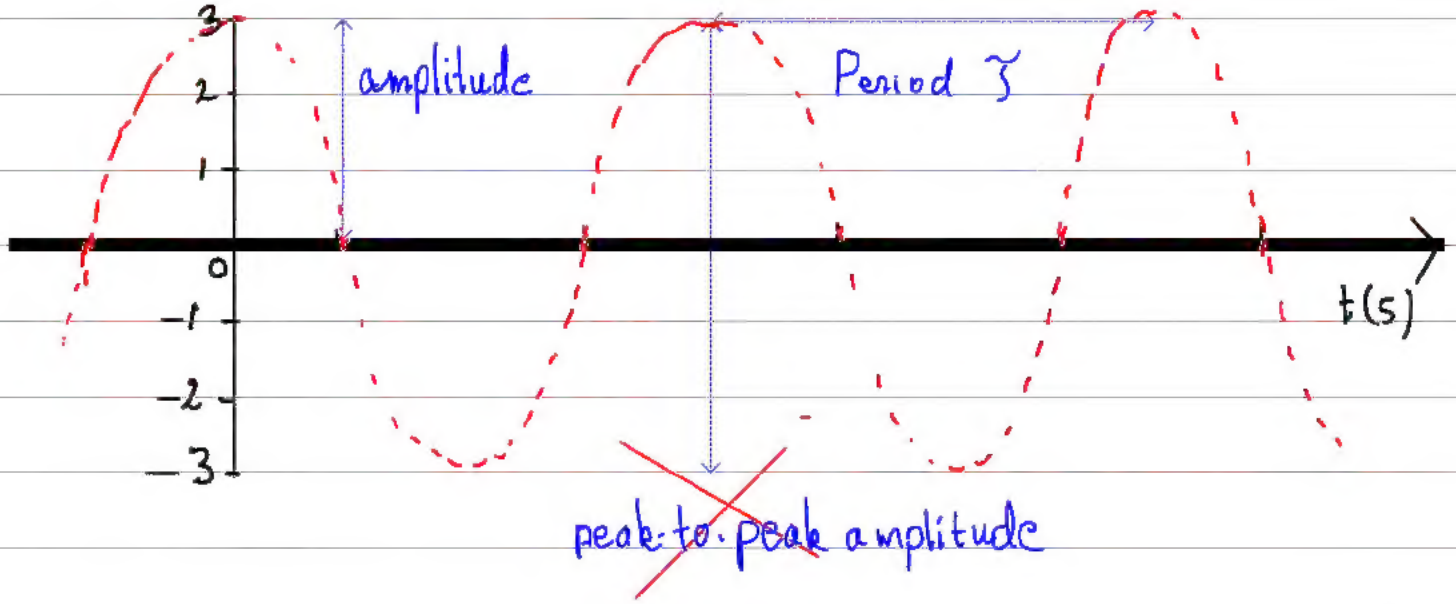
$\rightarrow \dot{x}(t) = \sqrt{2}\, C_3 \cos(\omega_n t + \phi)$

initial conditions: $\dot{x}(t=0) = 0 \rightarrow \phi = \pi/2$

$x(t=0) = 3\,m \qquad C_3 = 3\,m$

$\rightarrow \quad x(t) = 3 \sin(\sqrt{2}\, t + \pi/2)$ (S.I. units)

(or equivalently $x(t) = C_1 \sin \omega_n t + C_2 \cos(\sqrt{2}\, t)$ ($C_1 = 0$, $C_2 = 3$))



8/ Note that system amplitude is controlled by C_3 ↕

system timing is controlled by ϕ ↔

where C_3 & ϕ depend on initial conditions, i.e. excitation

9/ vibration natural frequency ω_n is determined from k, m:

$\omega_n = \sqrt{k/m}$ radians/s $= \sqrt{10/5} = \sqrt{2}\ s^{-1}$

$f = \omega_n / 2\pi$ = natural frequency (cycle/s) $= \sqrt{2}/2\pi = 0.225$ Hz

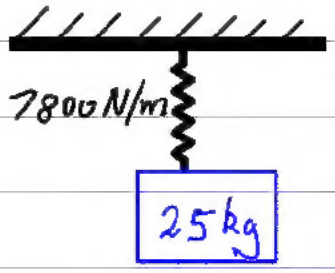
$\tau = 1/f = 2\pi/\omega_n = 1/0.225 = 4.4$ s

In our analysis of an engineering structure, our primary interest is often $\omega_n = \sqrt{\text{stiffness}/\text{mass}}$

How does a constant (static) force affect calculations.

E.g.

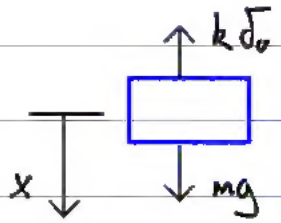
Find ω_n, ζ



1/ set origin where forces balance

$$\delta_0 = mg/k$$

2/ displace mass a small +ve distance x:



$$\rightarrow \quad \Sigma F = -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

3/ $\ddot{x} + \omega_n^2 x = 0$, where $\omega_n = \sqrt{k/m} = \sqrt{7800/25} = \sqrt{312}\ s^{-1}$

$$\zeta = 2\pi / \omega_n \qquad (\omega_n = \sqrt{312}) \qquad = 0.356\ s$$

If vibration amplitude = 30 mm, find maximum $|\vec{a}|$, $|\vec{v}|$.

$$\rightarrow \quad x(t) = C_3 \sin(\sqrt{312}\, t + \phi) \qquad (C_3 = 30\text{mm})$$

$$\rightarrow \quad \dot{x}(t) = (30\text{mm})\sqrt{312}\cos(\sqrt{312}\, t + \phi)$$

$$\rightarrow \quad \underline{\dot{X}_{max}} = (30\text{mm})\sqrt{312} \qquad 530\ \text{mm/s} \quad \text{or} \quad \boxed{0.53\ \text{m/s}}$$

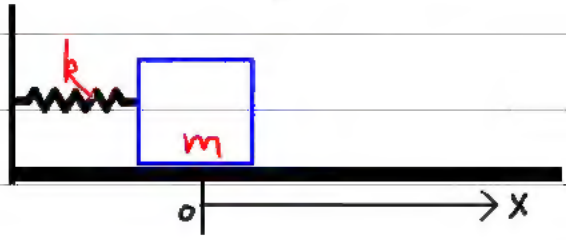
$$\ddot{x}(t) = (30\text{mm})\sqrt{312}\sqrt{312}\,(-1)\sin(\sqrt{312}\, t + \phi)$$

$$\rightarrow \quad \underline{\ddot{X}_{max}} = (30\text{mm})\sqrt{312}\sqrt{312} = 9360\ \text{mm/s}^2 \text{, or } \boxed{9.36\ \text{m/s}^2}$$

Summary

- Linear vibration model: in its simplest form,

$$\sum F_x = -kx = m\ddot{x}$$



- Solution is of the form:

- $x(t) = \boxed{C_3 \sin(\omega_n t + \phi)}$, $\omega_n = \sqrt{k/m}$
  = natural frequency

- Definitions of period τ, f (Hz), amplitude

Next Lecture

- More advanced examples of vibrations
  (Meriam 8/2)